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ADDENDUM

Addendum to 'On the finite difference between divergent sum and integral'

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Abstract. For the differences D in the title, a previous paper gave a general definition by the aid of cutoffs, but the methods given for evaluating D required fairly stringent conditions on summand and cutoff; it was thought that eventually these methods might become derivable as special cases from a more widely applicable argument operating with weaker restrictions. Here it is shown by a simple example that, contrary to such hopes, any plausible set of conditions that are significantly less restrictive than those used before fails to determine a unique value of D. Hence it seems unlikely that further worthwhile generalisations could be achieved.

This addendum is strictly a sequel to the earlier paper with the same title (Barton 1981, to be referred to as I), whose motivation, listing of references, notation and arguments will be taken as read. Here we discuss the mathematical status of the conditions needed to validate the prescriptions given in I for actually evaluating the differences D of the title, as opposed to merely defining them in principle. It will be shown, contrary to the expectations entertained when I was written, that these conditions cannot be significantly relaxed if a unique value of D is to be guaranteed from the outset. In other words, in I we envisaged for the future, a deeper and more widely applicable theory, from which the explicit prescriptions presented there would follow as special cases; but now it appears that no such theory is possible, and that by and large the earlier results are the best that can be achieved.

In I we presented three prescriptions; the two that remain in the running when partial sums and integrals are not given by familiar functions are the Abel–Plana (AP) and the Euler–Maclaurin (EM) prescriptions. The AP prescription requires fairly stringent analyticity and uniform-convergence conditions on the summand $f(n)$ and on the cutoff function $g(n|\lambda)$; the EM prescription replaces the analyticity by differentiability conditions. With f given and a suitable class of cutoffs to be discovered, one can run into three different kinds of problem, which we shall list in order of difficulty. Problems of the first and second kinds were met already in I, but are mentioned here in order to put the third kind into perspective. The purpose of this addendum is to show that in order to insure from the outset against the third kind of problem, namely against non-uniqueness, one needs conditions that cannot be significantly less restrictive than those needed for the AP or EM prescriptions.

We start from the fact that any successful prescription must produce an explicit functional $\mathscr{D}$ with the property

$$\lim_{\lambda\to\infty} \mathscr{D}\{f(n)g(n|\lambda)\} = \mathscr{D}\left\{\lim_{\lambda\to\infty} f(n)g(n|\lambda)\right\} = \mathscr{D}\{f(n)\} \equiv D, \qquad (1)$$

where it is essential not only that the functional $\mathscr{D}\{f(n)\}$ make sense, but also that the first limit in (1) should exist, be unique, and be equal to the second limit, for all g in the class of admissible cutoffs.

A snag of the first kind occurs if, armed with a given prescription for the functional $\mathscr{D}\{f\}$, one encounters a summand f for which $\mathscr{D}\{f\}$ diverges. For example, all versions of the AP prescription fail for the summand $f(n)=\exp(\alpha n)$ if $|\mathrm{Im}\,\alpha|>2\pi$. This can frustrate but not mislead one: the attempt fails to give any answer for D at all, and one must simply look for another prescription; in this example, our third method, namely the ε-averaging method, turns out to work.

A difficulty of the second kind occurs if the second limit in (1) exists (i.e. $\mathscr{D}\{f\}$ makes sense), but is not equal to the first limit because the first limit does not exist. For example, with the cutoff $g(n|\lambda)=\lambda^2/[(n-\lambda)^2+\gamma^2]$ which is forbidden by both the AP and the EM prescriptions, and with the trivial summand $f(n)=1$, $\mathscr{D}\{f(n)g(n|\lambda)\}$ continues to oscillate indefinitely as λ increases, even though $\mathscr{D}\{f(n)\}=0$ is well defined, and even though the cutoff obeys the basic permanence condition $\lim_{\lambda\to\infty}g(n|\lambda)=1$. Here, the mathematical method, so far from failing, in fact alerts one that with cutoffs like this the difference D is a physically ill defined quantity, for which no cutoff-independent value should have been sought in the first place.

The third kind of difficulty, and the only kind that might confuse even a careful calculation, arises if the first limit in (1) does exist, but depends on which cutoff one has chosen from a class of cutoffs all of which are *prima facie* reasonable. In other words the calculated value of D may fail to be unique without automatically signalling the fact. The AP and EM methods insure against just this possibility by restricting the admissible $g(n|\lambda)$ as explained earlier, and as explained in detail in I. At first sight one might suspect, as the present writer did, that these apparently rather narrow restrictions are dictated by the technical needs of the proofs rather than by the nature of the problem itself. In particular, at this stage one might still entertain the hope that if $\lim_{\lambda\to\infty}\mathscr{D}\{f(n)g(n|\lambda)\}$ exists at all, then the value it yields for D is unique subject only to the following physically plausible but mathematically much weaker conditions: (i) g is continuous and infinitely differentiable as a function of n; (ii) it obeys the permanence condition $\lim_{\lambda\to\infty}g(n|\lambda)=1$; (iii) for fixed n, $g(n|\lambda)$ increases towards 1 monotonically with increasing λ; (iv) for fixed λ, $g(n|\lambda)$ is non-increasing with increasing n. For instance, condition (iii) would rule out the pathological cutoff quoted above.

The point is that, disregarding subtleties, there appears to be no readily identifiable and plausible class of candidates for cutoff functions narrower than the class just specified yet significantly wider than those admitted by the AP or EM prescriptions. If this qualitative assessment is accepted, and if the class specified merely by the above conditions (i)–(iv) can be ruled out as too wide to guarantee uniqueness, then the AP- or EM-admissible cutoffs emerge as the natural choice, rather than as choices artificially restricted merely for convenience in establishing some explicit prescription.

An explicit example shows that the conditions (i)–(iv) are indeed too wide; the example is admittedly somewhat artificial, but the point is that it is perfectly admissible under the conditions. The only summand we consider is the trivial one $f(n)=1$, for which sensible prescriptions give $D=0$. The cutoffs will be constructed by modifying a discontinuous piecewise-constant descending-staircase function until it conforms to the conditions. We shall need an auxiliary function $E(x)$, monotonically decreasing, continuous and infinitely differentiable for all real x, with the following properties:

$$\begin{aligned} x &\leqslant -a/2: \qquad E(x)=1, \\ x &\geqslant a/2: \qquad E(x)=0, \end{aligned} \qquad (0<a<1). \tag{2}$$

Until further notice a is a fixed parameter. For example, we could adopt

$$-a/2 \leqslant x \leqslant x/2: \qquad E(x) = \exp\left[-\frac{1}{(a/2-x)}\exp\left(-\frac{1}{(a/2+x)}\right)\right]. \tag{3}$$

Next we chose a positive sequence $\{g_N(\lambda)\}$, $N = 0, 1, 2, \ldots$, such that

$$g_0(\lambda) = 1 \geqslant g_2(\lambda) \geqslant g_3(\lambda) > \ldots,$$

$$\lim_{N\to\infty} g_N(\lambda) = 0, \qquad \sum_{N=0}^{\infty} g_N(\lambda) \text{ converges}, \tag{4}$$

$$\lim_{\lambda\to\infty} g_N(\lambda) = 1.$$

For example we could chose $g_N(\lambda)$ to be $(1+N/\lambda)^{-p}$, or $[1+(N/\lambda)^p]^{-1}$, with $p > 1$. For each choice of $\{g_N\}$ we construct a cutoff function $g(n|\lambda)$ sketched in figure 1 and defined as follows:

$$0 \leqslant n \leqslant (1-a/2): \qquad g(n|\lambda) = g_0(\lambda) = 1, \tag{5a}$$

$$(N+a/2) \leqslant n \leqslant (N+1-a/2): \qquad g(n|\lambda) = g_N(\lambda), \tag{5b}$$

$$(N-a/2) \leqslant n \leqslant (N+a/2): \qquad g(n|\lambda) = \{g_{N-1}(\lambda)E(n-N) + g_N(\lambda)[1-E(n-N)]\}. \tag{5c}$$

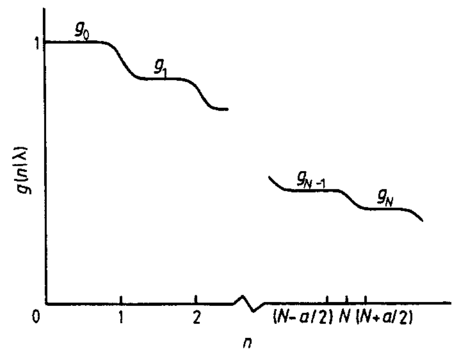

Figure 1. Sketch of a cutoff function defined by equations (3)–(5): not to scale. A typical element is shown on the right, as an aid in identifying the integrand in equation (8). The width of every curved portion is the same, namely a; but the height of the steps, namely $(g_{N-1}(\lambda) - g_N(\lambda))$, varies. Similar curves result from other choices of $E(x)$; notice that the curved portions can be highly asymmetric.

For integer values $n = N$, equations (5) imply

$$g(0|\lambda) = 1, \qquad g(N|\lambda) = \{g_N(\lambda) + [g_{N-1}(\lambda) - g_N(\lambda)]E(0)\} \qquad (N \geqslant 1). \tag{6}$$

We can now write down the sum $S(\lambda)$, the integral $I(\lambda)$, and their difference D. From (6) one finds immediately

$$S(\lambda) \equiv \tfrac{1}{2}g(0|\lambda) + \sum_{N=1}^{\infty} g(N|\lambda), \qquad S(\lambda) = \tfrac{1}{2} + E(0) + \sum_{N=1}^{\infty} g_N(\lambda). \tag{7}$$

Similarly one finds

$$I(\lambda) \equiv \int_0^\infty \mathrm{d}n\, g(n|\lambda),$$

$$I(\lambda) = \int_0^{(1-a/2)} \mathrm{d}n\, g_0(\lambda) + \sum_{N=1}^{\infty} \bigg(\int_{(N+a/2)}^{(N+1-a/2)} \mathrm{d}n\, g_N(\lambda)$$
$$+ \int_{(N-a/2)}^{(N+a/2)} \mathrm{d}n\, \{g_{N-1}(\lambda) E(n-N) + g_N(\lambda)[1 - E(n-N)]\} \bigg). \tag{8}$$

Changing each integration variable to $x = n - N$, rearranging, and again using $g(0|\lambda) = 1$, we obtain

$$I(\lambda) = (1 - a/2) + \int_{-a/2}^{a/2} \mathrm{d}x\, E(x) + \sum_{N=1}^{\infty} g_N(\lambda). \tag{9}$$

Equations (7) and (9) combine to give the end result

$$S(\lambda) - I(\lambda) = D = \bigg(-1/2 + a/2 + E(0) - \int_{-a/2}^{a/2} \mathrm{d}x\, E(x) \bigg). \tag{10}$$

(Since $S(\lambda) - I(\lambda)$ does not actually depend on λ, the final limit $\lambda \to \infty$ becomes redundant.)

It is evident from (10) that, given the function $E(x)$, the same value of D results from any cutoff constructed from a set $\{g_N(\lambda)\}$ in accordance with (4) and (5). But the crucial point is that different functions E yield different values of D, even though each such value is common to a wide class of cutoffs, as we have just seen. For example, even without changing the functional form of E, our point is made very simply by retaining the specification (3) and changing just the numerical value of a. The consequent change in D confirms that the conditions (i)–(iv) fail to determine D uniquely; our argument is then complete.

Reference

Barton G 1981 *J Phys. A: Math. Gen.* **14** 1009